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Scattering of aerodynamic noise by a semi-infinite compliant plate

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The acoustic scattering properties of a semi-infinite compliant plate immersed in turbulent flow are considered in the context of Lighthill's theory of aerodynamic noise. The turbulent eddies are replaced by a volume distribution of quadrupoles, and the reciprocal theorem used to transform the quadrupole scattering problem into one of the diffraction of a plane acoustic wave. This problem is solved by the Wiener–Hopf technique for the case when elastic forces in the plate are negligible, so that a local impedance condition relates the plate velocity to the pressure difference across the plate. Strong scattering of the near-field into propagating sound occurs when certain types of quadrupole lie sufficiently close to the plate edge, and we derive explicit expressions for the scattered fields in various cases. When fluid loading effects are small, and the plate relatively rigid, the results of Ffowcs Williams & Hall (1970) are recovered, in particular the U^5 law for radiated intensity. A quite different behaviour is found in the case of high fluid loading, when the plate appears to be relatively limp. The radiated intensity then increases with flow velocity U according to a U^6 law. In aeronautical situations, surface compliance is negligible in its effect on the scattering process, and the U^5 law must then apply provided the surface is sufficiently large. On the other hand, the effect of appreciable surface compliance is to greatly inhibit the near-field scattering from the surface edge. This weaker scattering is likely to be observed in underwater applications, where fluid loading effects are generally so high as to render unattainable the condition for a plate to be effectively rigid.

1. Introduction

According to the Lighthill (1952) theory of aerodynamic noise, the radiation from a region of turbulent flow may be calculated from the solution of a classical acoustic problem, in which the turbulent zone is replaced by a volume distribution of quadrupoles. The quadrupole strength $T_{ij}(\mathbf{x}, t)$ is assumed known in terms of properties of the turbulence alone, either from experiment or, for low Mach number flows, from the theory of incompressible turbulence. From his acoustic analogy, Lighthill was able to deduce his well-known U^8 law for increase of radiated intensity with flow velocity U, provided no boundaries are present in the flow. Curle (1955) was the first to consider the effects of rigid boundaries in the flow, showing that a rigid boundary could be replaced by a distribution of surface dipoles. A plausible dimensional estimate of the dipole strength leads to Curle's U^6 law for intensity. However, although Curle's solution of Lighthill's wave equation is formally exact, it does not in itself constitute a deductive theory of surface effects, for it requires a knowledge of dipole and quadrupole strengths independently. A deductive theory would determine the dipole strength when given only the quadrupole strength, and such a theory is badly needed in view of the paradoxes which have arisen in the past through the application of Curle's ideas to situations for which they were not intended. Fortunately, considerable progress has been made in recent years towards a clarification of various surface effects. Papers which determine the properties of the radiation entirely in terms of the Lighthill quadrupole stress tensor $T_{ij}(\mathbf{x}, t)$ are those by Powell (1960; the case of an infinite rigid plate), Ffowcs Williams (1965; the infinite flexible plate), Davies (1967; the rigid sphere), Crighton (1970; the composite boundary formed by two semi-infinite compliant plates of different properties), and Ffowcs Williams & Hall (1970; the rigid semi-infinite plate). A study of these papers reveals a variety of interesting and quite unexpected effects, and we have no reason to suspect the possibilities to be exhausted in the above references.

We are concerned here with developments of the work of Ffowcs Williams & Hall. They show that the scattering by a semi-infinite rigid plate is strongest for those quadrupoles which have both axes in planes with the plate edge as normal, as might be anticipated on physical grounds. The radiated pressure of such quadrupoles is enhanced over the free-field value by the factor $(k_0R)^{-\frac{3}{2}}$, k_0 being the acoustic wave-number, R the distance of the quadrupole from the edge. Thus the intensity produced by a quadrupole at a turbulence correlation length l_0 from the edge is increased, by the presence of the plate, by a factor M^{-3}, where M is the turbulence Mach number, $k_0 = Ml_0^{-1}$. In underwater applications, this effect is exceedingly important, if it applies there, for the maximum value of M which is ever encountered is of the order 10^{-2}. Such large effects deserve close attention, and are bound to arouse some controversy, so that the precise circumstances under which these effects occur are well worth examination. Accordingly, we shall consider here the problem of the scattering of the near-field of the quadrupole by a semi-infinite plate, with allowance for some flexibility of the plate. The results should be of particular interest in underwater situations, where the effects of fluid loading are always significant and where practical structures are often far from rigid. In this first study, however, we shall neglect elastic restoring forces in the plate, and shall treat the case of a locally reacting surface possessing inertia only. The case of an elastic plate or membrane would admittedly be less restrictive, but it appears that even the diffraction problem for the elastic plate has not yet received an entirely satisfactory treatment, despite several contributions to the literature. Subsequent work will attempt to remedy this situation, and also to consider other problems of scattering by sharp edges on various finite or semi-infinite bodies.

It may be useful here to give a brief outline of the method to be used. Ideally, we should like to obtain the Green's function for the problem, with source and observer in arbitrary positions. In the case of the rigid half-plane, the Green's function has been given in closed form by Macdonald (1915), and this function has been used by Ffowcs Williams & Hall (1970). For almost any other problem, it is virtually impossible to obtain the Green function in a useful form. Fortunately this proves to be no obstacle for our purposes, as we are only interested in the strong distant scattered field which arises when the source is closer than a wavelength, approximately, to the plate edge. By the reciprocal theorem, we may interchange source and observer, and solve for the field close to the edge due to a source at a large distance—i.e. due to a plane propagating incident wave. The Wiener–Hopf technique is the natural tool for such problems, and especially here, for the behaviour near the plate edge is determined by the behaviour of the Fourier transformed field for large values of the transformed variable. The latter behaviour is found very simply from the Wiener–Hopf method. Further, it is sufficient to work out the components of $\nabla\phi$ at a point near the edge and *in* the plane of the plate, ϕ denoting the velocity potential. It is found that $\nabla\phi$ has a singularity $O(x^{-\frac{1}{2}})$, while ϕ itself is $O(x^{\frac{1}{2}})$ near the edge, and therefore, near the edge the Helmholtz equation

$$(\nabla^2+k_0^2)\,\phi = 0$$

reduces to $\nabla^2\phi = 0$, and the flow is incompressible. General results for the form of ϕ in the incompressible flow past a sharp edge then allow us to determine the potential near the edge completely from a knowledge of $\nabla\phi$ at points in the plane of the plate. By reciprocity, the solution for $\nabla\phi$ will give us the field at infinity generated by a dipole near the edge, while a further differentiation of ϕ gives the distant field when a quadrupole lies close to the edge. This programme will be carried out in the subsequent sections.

The diffraction of a plane wave by a plate with an impedance boundary condition is analogous to the electromagnetic diffraction by a metallic half-plane of finite conductivity, a problem discussed by Senior (1952). The similarity is only formal, however, and the physical conditions of the two problems lead to quite different behaviour near the edge. In fact, Senior's paper contains an error, which appears to yield the same edge singularity as in the acoustic case; the error was first noticed by Kranzer & Radlow (1965), and they give the correct results. The formal similarity then persists only in the kernel of the Wiener–Hopf equation. It seems simpler here to derive results from first principles as required, rather than to attempt to manipulate the formulae of Senior into the necessary form. In particular, we give the derivation and solution of the Wiener–Hopf equation in some detail. This is perhaps advisable, as the method has found no previous application in aerodynamic noise theory though, as we shall see, it is particularly well suited to all problems of the scattering of the near-field of multipole sources by semi-infinite bodies.

2. The Wiener–Hopf problem

A plate of mass m per unit area and negligible thickness lies in the half-plane

$$x \geqslant 0, \quad y = 0, \quad -\infty < z < +\infty.$$

Elastic forces in the plate will be neglected, so that the response of the plate is governed by a purely local relation (pressure difference) = (specific mass) $\times$

(acceleration). The plate is surrounded by compressible fluid of density ρ_0 and sound speed a_0. We seek the acoustic field radiated to a distant point $\mathbf{x}_0$ when a monopole source is placed at an arbitrary point $\mathbf{x}$. By the reciprocal theorem (the proof of which, for the present circumstances, follows from a trivial application of Green's theorem), we may interchange emission and observation points, and solve for the field at a general point $\mathbf{x}$ due to the presence of a monopole at a distant point $\mathbf{x}_0$. Further, by suitable differentiations of the field with respect to $\mathbf{x}$, we can generate the field at $\mathbf{x}_0$ due to a multipole of arbitrary order at $\mathbf{x}$.

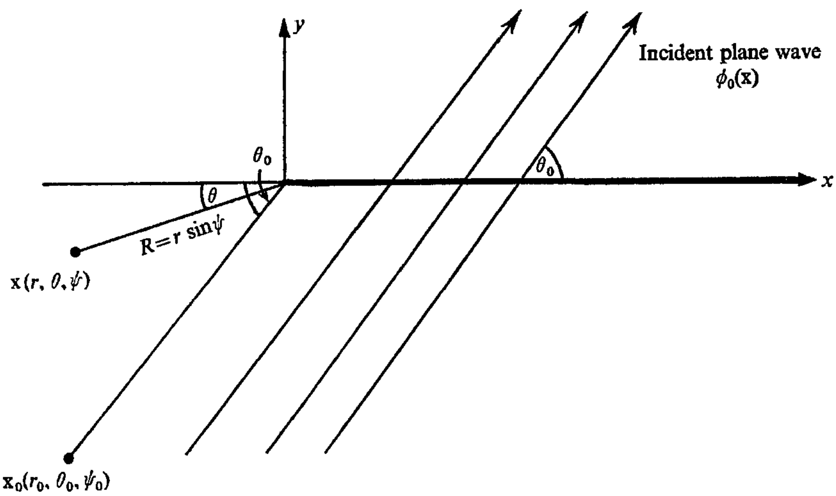

FIGURE 1. The co-ordinate system is such that the plate occupies the half-plane $x \geqslant 0$, $y = 0$, $-\infty < z < +\infty$. The points $\mathbf{x}$ and $\mathbf{x}_0$ have Cartesian components;

$$\mathbf{x} = -(r\sin\psi\cos\theta,\ r\sin\psi\sin\theta,\ r\cos\psi),$$

$$\mathbf{x}_0 = -(r_0\sin\psi_0\cos\theta_0,\ r_0\sin\psi_0\sin\theta_0,\ r_0\cos\psi_0),$$

and their projections upon the plane $z = 0$ are displayed.

We take a steady-state time factor $\exp[-i\omega t]$, $\omega > 0$, and work in terms of the velocity potential, denoting the incident field by ϕ_0, the scattered field by ϕ. It is convenient to introduce polar co-ordinates, as defined in figure 1, so that

$$\mathbf{x} = (-r\sin\psi\cos\theta,\ -r\sin\psi\sin\theta,\ -r\cos\psi).$$

Then the incident potential at $\mathbf{x}$ due to a monopole at $\mathbf{x}_0$ is

$$\phi_0(\mathbf{x}) = \frac{\exp[ik_0|\mathbf{x}-\mathbf{x}_0|]}{|\mathbf{x}-\mathbf{x}_0|},$$

$$\phi_0(\mathbf{x}) \sim \left(\frac{\exp[ik_0 r_0]}{r_0}\right)\exp[ik_0\sin\psi_0(x\cos\theta_0+y\sin\theta_0)+ik_0 z\cos\psi_0], \quad (2.1)$$

as $r_0\to\infty$, with r finite. Here $k_0 = \omega/a_0$ is the acoustic wave-number at frequency ω. It is evident that the factor

$$r_0^{-1}\exp[ik_0 r_0+ik_0 z\cos\psi_0]$$

is common to the whole field, and that this factor contains the whole of the variation with z. The factor will therefore be omitted, and restored at a later stage. It is then sufficient to take an incident field

$$\phi_0(x,y) = \exp[ik(x\cos\theta_0 + y\sin\theta_0)], \tag{2.2}$$

and a scattered field $\phi(x, y)$ satisfying

$$(\nabla^2 + k^2)\phi(x,y) = 0, \tag{2.3}$$

where k is the *reduced* wave-number $k_0 \sin\psi_0$.

We emphasize the simplicity of this approach. The distant field of an arbitrary three-dimensional multipole distribution can be obtained from the solution of a two-dimensional diffraction problem for an incident plane propagating wave. It will be seen later that only a small part of the information contained in the solution of the diffraction problem will be needed if we suppose the multipole to be close to the edge of the plate.

We proceed with the solution of the diffraction problem posed by (2.2) and (2.3). If p denotes the pressure in the scattered field, and v the plate velocity in the positive y direction, then $p = \rho_0 i\omega\phi$ and the condition on the plate is simply

$$p(x,0-) - p(x,0+) = -im\omega v(x),$$

which may be written in terms of the potential as

$$D(x) \equiv \phi(x,0+) - \phi(x,0-) = \frac{m}{\rho_0}\{\phi'(x,0) + ik\sin\theta_0 \exp(ikx\cos\theta_0)\} \tag{2.4}$$

for $x > 0$. The prime denotes the operation $\partial/\partial y$. At infinity, the scattered potential must satisfy a radiation or extinction condition.

We solve the system (2.2), (2.3), (2.4) by a straightforward application of D. S. Jones's technique (Noble 1958) for Wiener–Hopf problems. Define full and half-range Fourier transforms in x according to

$$\begin{aligned}\Phi(\alpha,y) &= \int_{-\infty}^{0} \phi(x,y)\, e^{i\alpha x}dx + \int_{0}^{\infty} \phi(x,y)\, e^{i\alpha x}dx \\ &\equiv \Phi_-(\alpha,y) + \Phi_+(\alpha,y),\end{aligned} \tag{2.5}$$

α being regarded as a complex variable. It is convenient to regard the wave-number k as complex also, introducing a small dissipative effect into the fluid which makes

$$k = k_1 + ik_2, \quad k_1, k_2 > 0.$$

Then it can be shown by the usual arguments in problems of this kind that $\Phi_+(\alpha, y)$ is a regular function of α in the upper half-plane

$$S_+(\operatorname{Im}\alpha > -k_2\cos\theta_0),$$

and that $\Phi_-(\alpha, y)$ is regular in the lower half-plane

$$S_-(\operatorname{Im}\alpha < k_2).$$

There is a strip $-k_2\cos\theta_0 < \operatorname{Im}\alpha < k_2$ in which the full-range transform exists as a regular function of α.

The transform applied to (2.3) gives

$$\left(\frac{\partial^2}{\partial y^2}-\gamma^2\right)\Phi(\alpha, y) = 0, \tag{2.6}$$

where $\gamma^2 = \alpha^2-k^2$. We define γ as that branch of $(\alpha^2-k^2)^{\frac{1}{2}}$ which tends to α as $\alpha \to \infty$ along the positive real axis. The α plane is to be cut from $+k$ to $k+i\infty$ in the first quadrant, and from $-k$ to $-k-i\infty$ in the third. Then it can be shown that $(\alpha^2-k^2)^{\frac{1}{2}} = -i(k^2-\alpha^2)^{\frac{1}{2}}$, where $(k^2-\alpha^2)^{\frac{1}{2}}$ is the branch which tends to $+k_1$ as α, $k_2 \to 0$. These are standard definitions and properties, to be found in Noble (1958). They show that the solution

$$\begin{aligned}\Phi(\alpha, y) &= A(\alpha)\exp[-\gamma y] \quad (y > 0)\\ &= B(\alpha)\exp[\gamma y] \quad (y < 0)\end{aligned} \tag{2.7}$$

to (2.6) satisfies either a radiation or an extinction condition as $|y| \to \infty$.

We now eliminate $A(\alpha)$, $B(\alpha)$ to obtain a linear relation between functions whose domains of regularity are known. It follows from the continuity of $\phi(x, y)$ across $y = 0$ for $x < 0$, that $\Phi_-(\alpha, y)$ is continuous across $y = 0$. Using this fact in subtracting the two forms of (2.7) for $y = 0+$ and $y = 0-$, we have

$$D_+(\alpha) \equiv \Phi_+(\alpha, 0+) - \Phi_+(\alpha, 0-) = A(\alpha) - B(\alpha).$$

Now we use the fact that $\phi'(x, y)$ is continuous across $y = 0$ for *all* x. This implies that $\Phi'(\alpha, y)$, $\Phi'_+(\alpha, y)$ and $\Phi'_-(\alpha, y)$ are each continuous across $y = 0$, and hence from (2.7) that

$$A(\alpha) = -B(\alpha).$$

We have then $D_+(\alpha) = 2A(\alpha)$, and

$$\Phi'_+(\alpha, 0) + \Phi'_-(\alpha, 0) = -\gamma A(\alpha) \quad \text{from} \quad (2.7).$$

Elimination of $A(\alpha)$ yields

$$\Phi'_+(\alpha, 0) + \Phi'_-(\alpha, 0) = -\tfrac{1}{2}\gamma D_+(\alpha). \tag{2.8}$$

We obtain a second linear relation of this kind by transforming the boundary condition (2.4) to give

$$D_+(\alpha) = \frac{m}{\rho_0}\left[\Phi'_+(\alpha, 0) - \frac{k\sin\theta_0}{\alpha + k\cos\theta_0}\right]. \tag{2.9}$$

Finally we eliminate $\Phi'_+(\alpha, 0)$ between (2.8) and (2.9) to obtain

$$\tfrac{1}{2}K(\alpha)D_+(\alpha) + \Phi'_-(\alpha, 0) = -\frac{k\sin\theta_0}{\alpha + k\cos\theta_0}, \tag{2.10}$$

in which $K(\alpha) = \mu + (\alpha^2-k^2)^{\frac{1}{2}}$, $\mu = 2\rho_0/m$. This is a standard form of Wiener–Hopf equation, i.e. a linear relation between one unknown function regular in S_+, one unknown function regular in S_-, and functions regular in the strip of overlap between S_+ and S_-.

It can be shown that $K(\alpha)$ has no zeros in the strip when the α plane is cut as stated earlier. We may therefore write

$$K(\alpha) \equiv K_+(\alpha)K_-(\alpha),$$

where the factors $K_\pm(\alpha)$ are regular and non-zero in $S_\pm$ respectively. Explicit forms for the factors are given in the appendix. Then (2.10) becomes

$$\tfrac{1}{2}K_+(\alpha)D_+(\alpha)+\frac{\Phi'_-(\alpha,0)}{K_-(\alpha)}=-\frac{k\sin\theta_0}{K_-(\alpha)(\alpha+k\cos\theta_0)},$$

and the next step is to write

$$L(\alpha)\equiv-\frac{k\sin\theta_0}{K_-(\alpha)(\alpha+k\cos\theta_0)}\equiv L_+(\alpha)+L_-(\alpha),$$

a sum of functions regular in $S_\pm$ respectively. The required splitting is trivial, giving

$$\left.\begin{aligned} L_-(\alpha)&=-\frac{k\sin\theta_0}{\alpha+k\cos\theta_0}\left(\frac{1}{K_-(\alpha)}-\frac{1}{K_-(-k\cos\theta_0)}\right),\\ L_+(\alpha)&=-\frac{k\sin\theta_0}{(\alpha+k\cos\theta_0)K_-(-k\cos\theta_0)}.\end{aligned}\right\}\tag{2.11}$$

We now have
$$\begin{aligned}\tfrac{1}{2}K_+(\alpha)D_+(\alpha)-L_+(\alpha)&=L_-(\alpha)-\frac{\Phi'_-(\alpha,0)}{K_-(\alpha)}\\ &=J(\alpha)\quad\text{say.}\end{aligned}\tag{2.12}$$

This equation holds only in the common strip of regularity of both sides. However, the left side defines $J(\alpha)$ throughout S_+, while the right side defines $J(\alpha)$ throughout S_-. Because of the common strip of regularity, each side of (2.12) provides the analytic continuation of the other, and the composite function $J(\alpha)$ defined by (2.12) is regular in the entire α plane. Provided that we can show that both sides of (2.12) have only algebraic growth as $\alpha\to\infty$ in all directions, it follows from the extension of Liouville's theorem that $J(\alpha)$ must be a polynomial in α. In fact we show here that $J(\alpha)$ must vanish at infinity, and hence that $J(\alpha)$ vanishes identically.

To do this, consider the potential difference

$$D(x)=\phi(x,0+)-\phi(x,0-).$$

This is a continuous function of x, which is identically zero for $x<0$, so that we must have
$$D(x)\sim x^\lambda\quad\text{as}\quad x\to0+\quad\text{with}\quad\lambda>0.$$

The behaviour of the transform $D_+(\alpha)$ for large $|\alpha|$ is found from the Abel theorem, that

$$D_+(\alpha)\sim\int_0^\infty x^\lambda e^{i\alpha x}dx\sim|\alpha|^{-\lambda-1},\tag{2.13}$$

as $\alpha\to\infty$ with $\operatorname{Im}\alpha>0$. Similarly, we have that

$$\Phi'_-(\alpha,0)\sim\int_{-\infty}^0\phi'(x\to0-,0)e^{i\alpha x}dx$$

as $\alpha\to\infty$ with $\operatorname{Im}\alpha<0$. The integrand here is proportional to the fluid velocity induced by the scattered wave, and on physical grounds it cannot have a non-integrable singularity at $(0,0)$. Thus we must have $\phi'(x,0)\sim(-x)^{-\nu}$ as $x\to0-$ with $\nu<1$, and this implies that

$$\Phi'_-(\alpha,0)\sim|\alpha|^{\nu-1},\quad\alpha\to\infty,\quad\operatorname{Im}\alpha<0.\tag{2.14}$$

Note that these edge conditions are much weaker than those often used (Noble 1958). From the appendix we have that

$$K_{\pm}(\alpha) \sim |\alpha|^{\frac{1}{2}} \quad \text{as} \quad |\alpha| \to \infty$$

in appropriate half-planes, while from (2.11) we have that $L_{\pm}(\alpha)$ are each $O(|\alpha|^{-1})$. With these estimates and the edge conditions (2.13), (2.14), it is easily established that the function $J(\alpha)$ is identically zero.

We thus determine $\Phi'_{-}(\alpha, 0)$ as

$$\Phi'_{-}(\alpha, 0) = -\frac{k \sin\theta_0}{\alpha + k\cos\theta_0}\left(1 - \frac{K_{-}(\alpha)}{K_{-}(-k\cos\theta_0)}\right). \tag{2.15}$$

We shall now suppose that, in the source–excitation problem, the source is very close to the edge of the plate. In the reciprocal problem, this corresponds to observation of the scattered field at a point $\mathbf{x}$ very close to the edge and hence, by Fourier inversion and use of the Abel theorem, to evaluation of (2.15) in the limit $|\alpha| \to \infty$. The result will be true in a strict asymptotic sense as $|\mathbf{x}| \to 0$, but some care will be needed in its application and interpretation. Consider first the case of a surface which is nearly rigid in the sense that $\mu/k \ll 1$. Then the criterion for the first term of the asymptotic development of (2.15) to dominate is that $k|\mathbf{x}| \lesssim 1$, which holds within a wavelength of the edge. On the other hand, when the fluid loading is high and the surface nearly limp, the requirement is presumably much more stringent, and we probably need $\mu|\mathbf{x}| \lesssim 1$ (though this is difficult to see in view of the complexity of $K_{-}(\alpha)$). In the application to the aerodynamic noise problem we shall make the identification $k = Ml_0^{-1}$, $\mu = \epsilon l_0^{-1}$, where l_0 is a turbulence eddy length scale, $M = u_0/a_0$ the turbulence Mach number and ϵ the fluid loading parameter $2\rho_0 l_0/m$. Aeronautical applications generally involve only small values of μ/k, so that our results will be applicable to the scattering of the near-field of multipoles closer essentially than a wavelength to the edge. In underwater situations, the Mach number is never greater than 10^{-2}, while ϵ is still only of order unity at most. It is, therefore, still possible to satisfy the condition $\mu|\mathbf{x}| \lesssim 1$ with a value of $|\mathbf{x}|$ of order l_0. The point is that when the medium is water, the wavelength is so large that the condition on $|\mathbf{x}|$ can be satisfied with a realistic value of order l_0, even though the surface is fairly limp in the sense that $\mu/k \gg 1$. It is important to note that a decrease of $|\mathbf{x}|$ below a value of order l_0 is irrelevant, in view of the crude way in which we are at present obliged to model the turbulence structure in relation to problems of flow-induced sound and vibration.

3. The field in the vicinity of the edge

For points sufficiently close to the edge, the discussion of the previous paragraph justifies the use of the asymptotic form of (2.15), namely

$$\Phi'_{-}(\alpha, 0) \sim \frac{k \sin\theta_0}{K_{-}(-k\cos\theta_0)}\,\alpha^{-\frac{1}{2}}. \tag{3.1}$$

Because of the linearity of the problem, it is sufficient, and convenient, to take

$$\Phi'_-(\alpha, 0) \sim \alpha^{-\frac{1}{2}} e^{-\frac{1}{4}\pi i},$$

and use of the Abel theorem as in (2.14) then gives

$$\frac{\partial \phi}{\partial y}(x, 0) \sim \frac{1}{(-\pi x)^{\frac{1}{2}}} \quad \text{as} \quad x \to 0-. \tag{3.2}$$

We have also that
$$\frac{\partial \phi}{\partial x}(x, 0) = 0 \quad \text{for} \quad x < 0, \tag{3.3}$$

a result which follows from the fact that $\phi(x, y)$ is an odd function of y which is continuous across $y = 0$ when $x < 0$. We can use the results (3.2) and (3.3) to evaluate the potential at a general point (x, y) near to the plate edge. For the singularity of $\partial\phi/\partial y$ near the edge implies that the motion there is effectively incompressible, and the solution (3.2) shows that the general form for the dominant term of the potential must be

$$\phi(R, \theta) = AR^{\frac{1}{2}} \cos \tfrac{1}{2}\theta + BR^{\frac{1}{2}} \sin \tfrac{1}{2}\theta, \tag{3.4}$$

where $x = -R\cos\theta$, $y = -R\sin\theta$ and $R = r\sin\psi$ is distance from the edge, in terms of the polar co-ordinates defined earlier. In the case of a rigid surface, of course, the condition of zero velocity on $y = 0$, $x > 0$ can be satisfied by an expression of the form (3.4) without any higher-order terms. When the surface is not rigid, higher-order terms are needed for the complete solution, but the terms quoted in (3.4) must dominate when R is sufficiently small, the conditions being that $kR \lesssim 1$ when $\mu/k \ll 1$ while $\mu R \lesssim 1$ when $\mu/k \gg 1$.

The constants A, B can be determined from (3.2) and (3.3). We find that

$$\phi = -\frac{2R^{\frac{1}{2}} k_0 \sin\theta_0 \sin\psi_0 \sin\frac{1}{2}\theta}{\pi^{\frac{1}{2}} K_-(-k_0 \sin\psi_0 \cos\theta_0)} \frac{\exp[i(k_0 r_0 + k_0 z\cos\psi_0 + \frac{1}{4}\pi)]}{r_0}, \tag{3.5}$$

where all constants omitted at various stages have now been restored. We now regard ϕ as a function of $\mathbf{x}_0$, in which case the reciprocal theorem shows that ϕ is the scattered potential at $\mathbf{x}_0$ due to a monopole of unit strength at $\mathbf{x}$. Differentiations of ϕ with respect to $\mathbf{x}$ will give us the potential at $\mathbf{x}_0$ due to multipoles of various orders at $\mathbf{x}$. For example, $\partial\phi/\partial x$ is the potential at $\mathbf{x}_0$ due to a dipole of unit strength at $\mathbf{x}$ with axis in the $+x$ direction; $\partial^2\phi/\partial x\,\partial y$ gives the potential at $\mathbf{x}_0$ due to a quadrupole at $\mathbf{x}$ with one axis in the $+x$ direction, the other in the $+y$ direction. We shall not examine all the possibilities, but list below the potentials due to various multipoles of interest

$$\frac{\partial\phi}{\partial x} = \frac{k_0 \sin\theta_0 \sin\psi_0 \sin\frac{1}{2}\theta}{\pi^{\frac{1}{2}} R^{\frac{1}{2}} K_-(-k_0 \sin\psi_0 \cos\theta_0)} \frac{\exp[i(k_0 r_0 + k_0 z\cos\psi_0 + \frac{1}{4}\pi)]}{r_0}, \tag{3.6}$$

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{k_0 \sin\theta_0 \sin\psi_0 \cos\frac{3}{2}\theta}{2\pi^{\frac{1}{2}} R^{\frac{3}{2}} K_-(-k_0 \sin\psi_0 \cos\theta_0)} \frac{\exp[i(k_0 r_0 + k_0 z\cos\psi_0 + \frac{1}{4}\pi)]}{r_0}, \tag{3.7}$$

$$\frac{\partial^2\phi}{\partial x\,\partial z} = -\frac{ik_0^2 \sin\theta_0 \sin\psi_0 \cos\psi_0 \sin\frac{1}{2}\theta}{\pi^{\frac{1}{2}} R^{\frac{1}{2}} K_-(-k_0 \sin\psi_0 \cos\theta_0)} \frac{\exp[i(k_0 r_0 + k_0 z\cos\psi_0 + \frac{1}{4}\pi)]}{r_0}, \tag{3.8}$$

$$\frac{\partial^2\phi}{\partial z^2} = \frac{2R^{\frac{1}{2}}k_0^3\sin\theta_0\sin\psi_0\cos^2\psi_0\sin\frac{1}{2}\theta}{\pi^{\frac{1}{2}}K_-(-k_0\sin\psi_0\cos\theta_0)}\frac{\exp[i(k_0r_0+k_0z\cos\psi_0+\frac{1}{4}\pi)]}{r_0}. \tag{3.9}$$

For comparison we need also the distant fields emitted by the same multipoles in the absence of the plate. These are obtained directly from (2.1), and are typified by

$$\frac{\partial^2\phi_0}{\partial x\,\partial y} = -k_0^2\sin\psi_0\sin\theta_0\cos\theta_0\frac{\exp[i(k_0r_0+k_0z\cos\psi_0)]}{r_0}, \tag{3.10}$$

provided $k_0R \ll 1$. The fields of other quadrupoles differ from (3.10) only in the directivity factors. This completes the derivation of detailed expressions for the distant fields of multipoles close to the edge of the plate. The application of these expressions to the scattering of the near-field of aerodynamic noise sources follows in the next section.

4. Scattering of aerodynamic noise

The Lighthill (1952) theory of aerodynamic noise involves the solution of a classical radiation problem, through the formulation of an inhomogeneous wave equation for the fluid density. The inhomogeneity takes the form of a double space derivative, $\partial^2T_{ij}/\partial x_i\,\partial x_j$, representing the acoustic effect of a volume distribution of quadrupoles of strength $T_{ij}(\mathbf{x},t)$. Lighthill shows how T_{ij} can be regarded as known in terms of properties of the turbulent flow whose acoustic output is required, at any rate provided the turbulence Mach number is low. Here we need little knowledge of T_{ij} beyond the idea that it is a function dominated by the energy-containing eddies of the turbulence. These are characterized by the r.m.s. velocity u_0, and an integral correlation length l_0. (Effects of turbulent eddy convection by a mean flow are ignored here.) Thus we can associate a characteristic frequency u_0/l_0 with T_{ij}, and provided the Mach number $M = u_0/a_0$ is low this will also be the typical frequency of the emitted radiation. The acoustic wavenumber is therefore given by $k_0 = Ml_0^{-1}$, and this is the only connexion between the acoustic and hydrodynamic aspects of the problem which we need in order to predict the increase in the power output of a *given* quadrupole which is caused by the presence of the half-plane.

Consider first the relative fields of the (x,y), (x,z) and (z,z) quadrupoles, as given by (3.7), (3.8) and (3.9). Apart from constants and directivity factors of order unity, the ratios between those fields are as

$$1 : k_0R : (k_0R)^2, \quad \text{respectively.}$$

The formulae have been derived under the assumption that $k_0R \ll 1$, and we see therefore that quadrupoles with both axes in the plane with the plate edge as normal produce the most powerful scattered field. Those with one axis in that plane and the other parallel to the plate edge generate a linear field less powerful by a factor k_0R, those with both axes parallel to the edge by a factor $(k_0R)^2$. These results are independent of the function K_-, and we expect them to hold quite generally in problems of edge scattering.

We confine attention now to the quadrupoles with both axes in the plane normal to the plate edge. If the plate is perfectly rigid, we have from (A 7),

$$K_{-}(-k\cos\theta_0) = -i(2k)^{\frac{1}{2}}\cos\tfrac{1}{2}\theta_0,$$

and then from (3.7) the distant field of an (x, y) quadrupole close to the edge is given by

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{ik_0^{\frac{1}{2}}\sin\frac{1}{2}\theta_0\sin^{\frac{1}{2}}\psi_0\cos\frac{3}{2}\theta}{(2\pi)^{\frac{1}{2}}R^{\frac{3}{2}}}\,\frac{\exp[i(k_0r_0+k_0z\cos\psi_0+\frac{1}{4}\pi)]}{r_0}. \tag{4.1}$$

A first-order correction to this result, allowing for slight compliance of the plate, can be obtained by using the full expression (A 7). The expression (4.1) agrees with a result of Ffowcs Williams & Hall (1970) if some changes of notation are made; in particular, Ffowcs Williams & Hall use r_0 for distance from the edge where we use R, and θ for the far-field angle in place of our θ_0. The ratio of scattered amplitude to direct amplitude is found from (4.1) and (3.10) as

$$\beta \sim (k_0R)^{-\frac{3}{2}} = (l_0/R)^{\frac{3}{2}}M^{-\frac{3}{2}}. \tag{4.2}$$

This amplification factor is the central result of the work of Ffowcs Williams & Hall. It may be applied to quadrupoles closer than a wavelength k_0^{-1} to the plate edge and shows, for example, that the power output of a *given* quadrupole at $R \sim l_0$ is increased by a factor M^{-3} by the presence of the plate. In underwater flow noise M never exceeds 10^{-2} (even when based on mean velocity, rather than r.m.s. velocity), and this scattering effect increases the power output of quadrupoles close to the edge by at least 60 dB. The power output of the quadrupoles in free-field varies as u_0^8 (Lighthill 1952), and therefore the scattered power varies as u_0^5—a result which could hardly be expected from any application of dimensional analysis to Curle's (1955) solution of Lighthill's equation.

We obtain a quite different result by considering the case of a relatively limp surface, in which $\mu/k_0 \gg 1$. Using the result (A 12) we find the distant field of an (x, y) quadrupole to be

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{ik_0\sin\theta_0\sin\psi_0\cos\frac{3}{2}\theta}{2\pi^{\frac{1}{2}}R^{\frac{3}{2}}\mu^{\frac{1}{2}}}\,\frac{\exp[i(k_0r_0+k_0z\cos\psi_0)]}{r_0}. \tag{4.3}$$

The amplification factor is now

$$\beta \sim k_0^{-1}R^{-\frac{3}{2}}\mu^{-\frac{1}{2}} = (l_0/R)^{\frac{3}{2}}\epsilon^{-\frac{1}{2}}M^{-1}, \tag{4.4}$$

where ϵ is the fluid loading parameter, $\epsilon = 2\rho_0 l_0/m$. This gives the asymptotic form of β for a fixed value of ϵ as $M \to 0$, whereas (4.2) gives the asymptotic form for a fixed M as $\epsilon \to 0$. A point to note about (4.3) is that it shows that the scattered field vanishes as the surface mass m tends to zero. This is required physically, for then the surface would move with the flow generated by the incident field, and no scattering would occur.

Supposing that (4.4) holds for $M = 10^{-2}$ and $\epsilon = O(1)$, the power output of quadrupoles near to the plate edge would be increased by about 40 dB, as against the 60 dB which would be predicted by (4.2). The scattered power now varies as u_0^6/ϵ, in place of the u_0^5 law of Ffowcs Williams & Hall (1970). That law considerably overestimates the scattered power if applied to circumstances in which the plate is not rigid relative to the surrounding fluid. A similar inhibition of the scattered power through the effect of high fluid loading ($\epsilon/M \gg 1$) was found in a previous study (Crighton 1970). There the scattering surface was taken to be formed by two joined semi-infinite planes with different specific masses. A u_0^4 law was found for the case of low fluid loading ($\epsilon/M \ll 1$), whereas a u_0^6 law applies in the other limit $\epsilon/M \gg 1$.

The condition for the plate to be effectively rigid in the present context is that $\epsilon/M \ll 1$. In aeronautical situations this condition is generally well satisfied, the smallness of ρ_0 leading to a small value of ϵ, with M being not too small. On the other hand, underwater applications seem usually to involve values of ϵ of order unity, with extremely small Mach numbers. The condition $\epsilon/M \gg 1$ is strongly satisfied, the mass of fluid contained within a wavelength on either side of the plate being much greater than the mass of the plate. It therefore appears that the amplification factor (4.2) applies in aeronautical contexts, while (4.4) holds for underwater flow-noise scattering. There is no value in regarding the u_0^6 law as due to a distribution of surface dipoles in the sense of Curle (1955). For the surface is not compact relative to the emitted wavelength, and the function K_- appearing in (3.5) is clearly not an analytic function of wave-number k_0, and both of these facts preclude any such simple interpretation. Use of the complete expression (A 11) for K_- would yield an astonishing dependence upon M, which could not possibly be explained in elementary multipole terms.

It is possible to generalize our results to the scattering of aerodynamic noise by a wedge of arbitrary angle. As an example, we take the case of a wedge of internal angle $\frac{1}{2}\pi$. The essence of our method lies in the examination of the potential near to the edge when the incident field is a plane propagating acoustic wave. The behaviour near the edge may be obtained simply by dimensional reasoning, at least when the faces of the wedge are rigid and k_0^{-1} is the only length scale in the problem. For the scattered potential is dimensionless, and must tend to the incompressible flow potential past a right-angled wedge as $k_0 R \to 0$, where R is distance from the edge. It follows that

$$\phi \sim (k_0 R)^{\frac{2}{3}} \quad \text{as} \quad k_0 R \to 0,$$

and that

$$\frac{\partial^2 \phi}{\partial x \, \partial y} \sim k_0^{\frac{2}{3}} R^{-\frac{4}{3}}.$$

The amplification factor is

$$\beta \sim (l_0/R)^{\frac{4}{3}} M^{-\frac{4}{3}}, \tag{4.5}$$

and the scattered power output varies as $u_0^{\frac{16}{3}}$.

We expect this result to apply to nearly rigid wedges, as before. If, however, the faces of the wedge were to respond to the applied pressure field in the same sort of way as the plate considered earlier, there would be two length scales in the problem, μ^{-1} and k_0^{-1}. In the case $\mu/k_0 \gg 1$, the form (4.3) might lead us to expect that ϕ will vary as k_0, the remainder of the dependence being taken up by the parameter μ, so that

$$\phi \sim (k_0/\mu)(\mu R)^{\frac{2}{3}}. \tag{4.6}$$

This form would again lead us to a u_0^6 law for scattered power, though its validity is no more than suggested by the preceding argument.

5. Conclusions

Lighthill's acoustic analogy for aerodynamic noise problems involves the solution of a wave equation with a quadrupole inhomogeneity which is assumed known. When boundaries are present in the flow, the formal solution to Lighthill's equation is supplemented by surface integrals involving quantities other than the quadrupole strength, and thus constitutes no more than an integral equation for the field. The proper deductions about the effects of solid surfaces cannot be drawn until the integral equation is solved.

We have been concerned here with the solution of the integral equation in the case when the surface is formed by a semi-infinite compliant plate (though in fact we solve the differential form of the wave equation). We use the term compliant to imply that the plate is capable of deflexion, but with a purely local response to the fluctuating pressure field upon it. This is the case if the plate possesses inertia, but negligible elastic resistance to deformation, so that the usual differential relation for the plate response reduces to a simple local proportionality between deflexion and pressure difference across the plate. We regard the present work as a first step towards a treatment of the more difficult problem of the elastic plate, though the adoption of a local impedance condition has great relevance to marine applications, where effects of elasticity are usually much smaller than those of inertia.

It is found that a strong scattering of the near-field of a quadrupole into propagating acoustic energy occurs when the quadrupole distance R from the edge is much smaller than a wavelength k_0^{-1}, provided both axes of the quadrupole lie in the plane with plate edge as normal. The amplitude of the scattered wave depends upon a fluid loading parameter ϵ, and upon the wave-number k_0—or equivalently, upon the turbulence Mach number M. A complete solution is given for the dependence of the scattered field upon ϵ, M and R, and upon angular factors specifying the direction in the far-field and the position of the quadrupole.

The results of Ffowcs Williams & Hall (1970) for the rigid plate are recovered by setting $\epsilon = 0$. The principal result is then contained in the formula (4.2) for the amplification factor, which shows that the scattering is extremely powerful and will increase the power output of a given quadrupole at $R \sim l_0$ by 60 dB at a Mach number 10^{-2}. The details of the scattering have been fully discussed by Ffowcs Williams & Hall. The only new feature of our work in this connexion is the derivation of the first-order correction away from perfect rigidity, obtained by inserting (A 7) into (3.5).

A new result is found when the plate is limp relative to the surrounding fluid. It is found then that a much weaker type of scattering occurs, in which the u_0^5 law predicted by (4.2) for total acoustic power is replaced by a dependence upon u_0^6/ϵ. We suggest that the u_0^5 law applies in many aeronautical problems, but that the u_0^6 dependence replaces it in underwater situations where fluid loading effects are always very great.

Now the above remarks appear to contradict some results and conjectures of Ffowcs Williams & Hall (1970), who have also treated the so-called 'acoustically soft' half-plane, and have shown that the results are essentially the same as for the rigid half-plane, apart from changes in directivity factors. They suggest that cases of intermediate compliance will probably not lead to any very different results. It is worth looking briefly at just what is involved here. Let [ ] denote the discontinuity in any quantity across the half-plane. Then the problem which we have solved is characterized by conditions of the form $[\phi] = \eta\, \partial\phi/\partial y$, $[\partial\phi/\partial y] = 0$, and always leads to an integrable singularity in $\nabla\phi$ near the plate edge. Our results for a limp surface are obtained by taking the limit $\eta \to 0$ of the solution of the complete problem, and continuity of normal velocity is retained in this limit. There is a second class of problems, however, characterized by the conditions $\eta[\partial\phi/\partial y] = \phi$, $[\phi] = 0$, which do *not* involve any edge singularity in $\nabla\phi$ *except* in the isolated case $\eta = 0$, which is referred to as the 'acoustically soft' case. Thus the soft case is clearly an exception, and both types of intermediate compliance are likely to lead to new results. The second class of boundary conditions does not seem to have any physical relevance in acoustic problems (though it does in electromagnetic diffraction) and we have considered only the first class. A physically meaningful 'limp' surface can be obtained as a limit of the first class, and as we have seen, scatters a fundamentally weaker field than the rigid surface.

A point to note about our work is that the representation of the turbulence by *quadrupole* sources is *quite essential.* Ribner's (1962) theory provides a viable alternative to the Lighthill model when boundaries are absent, and involves a weak $O(M^2)$ monopole distribution in place of Lighthill's $O(1)$ quadrupoles. However, such a theory fails to emphasize the strength of near-field relative to far-field in the way that is crucial in this scattering theory. The use of an $O(M^2)$ monopole in place of an $O(1)$ quadrupole results, in fact, in an underestimate of the scattered field by a factor $M^2(R/l_0)^2$ as can be seen from (3.5) and (3.7). For the same reason, of course, the isotropic part $\delta_{ij}T_{kk}/3$ of T_{ij} scatters a very much weaker field than the non-isotropic quadrupole elements.

Further work is now needed to determine whether the present conclusions are drastically altered by a variety of effects not yet treated. We mention the possible effects of a mean flow, of finite size of the plate, finite edge thickness and curvature, and viscous action near the edge. Work is at present in hand on these effects, in the hope that this will lead to a satisfactory understanding of the general features of the scattering by sharp-edged bodies.

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Appendix

We require the multiplicative decomposition of

$$K(\alpha) = \mu + (\alpha^2 - k^2)^{\frac{1}{2}}. \tag{A 1}$$

The complete factorization is given below, but we prefer to start with an asymptotic factorization as $\mu \to 0$, regarding the function μ as a perturbation of the basic kernel $(\alpha^2-k^2)^{\frac{1}{2}}$ which alone is present in the case of a rigid surface. Since the perturbation grows less rapidly than the basic kernel as $|\alpha|\to\infty$, we can assert from the work of Kranzer & Radlow (1962, 1965) that the required factorization will take the form

$$K_\pm(\alpha) = (\alpha\pm k)^{\frac{1}{2}}+\mu M_\pm(\alpha)+o(\mu). \tag{A 2}$$

Inserting these forms into (A 1) and letting $\mu\to 0$, we require the unknown functions $M_\pm(\alpha)$ to satisfy

$$\frac{M_+(\alpha)}{(\alpha+k)^{\frac{1}{2}}}+\frac{M_-(\alpha)}{(\alpha-k)^{\frac{1}{2}}} = \frac{1}{(\alpha^2-k^2)^{\frac{1}{2}}}. \tag{A 3}$$

The additive decomposition of $(\alpha^2-k^2)^{-\frac{1}{2}}$ is well-known (Noble 1958, p. 21) in the form

$$\begin{aligned}\frac{1}{(\alpha^2-k^2)^{\frac{1}{2}}} &= \frac{1}{\pi(\alpha^2-k^2)^{\frac{1}{2}}}\left\{\cos^{-1}\left(\frac{\alpha}{k}\right)+\cos^{-1}\left(-\frac{\alpha}{k}\right)\right\}\\ &= P_+(\alpha)+P_-(\alpha) \quad \text{say.}\end{aligned} \tag{A 4}$$

From (A 3) and (A 4) we find, for example, that

$$K_-(\alpha) = (\alpha-k)^{\frac{1}{2}}\left\{1+\frac{\mu}{\pi(\alpha^2-k^2)^{\frac{1}{2}}}\cos^{-1}\left(-\frac{\alpha}{k}\right)+o(\mu)\right\}. \tag{A 5}$$

For the work of §2 we require

$$K_-(\alpha)\sim\alpha^{\frac{1}{2}} \quad \text{as} \quad \alpha\to\infty, \quad \operatorname{Im}\alpha<0, \tag{A 6}$$

$$K_-(-k\cos\theta_0) = -i(2k)^{\frac{1}{2}}\cos\left(\tfrac{1}{2}\theta_0\right)\left\{1+\frac{i\mu\theta_0}{\pi k\sin\theta_0}+o(\mu)\right\}. \tag{A 7}$$

Next we derive a complete factorization. By logarithmic differentiation of (A 1) we have

$$\begin{aligned}\frac{K_+'}{K_+}+\frac{K_-'}{K_-} &= \frac{1}{\mu+(\alpha^2-k^2)^{\frac{1}{2}}}\frac{\alpha}{(\alpha^2-k^2)^{\frac{1}{2}}}\\ &= \frac{1}{2}\left(\frac{1}{\alpha-p}+\frac{1}{\alpha+p}\right)-\frac{\mu}{2(\alpha^2-k^2)^{\frac{1}{2}}}\left(\frac{1}{\alpha-p}+\frac{1}{\alpha+p}\right).\end{aligned} \tag{A 8}$$

Here we have multiplied through by $\mu-(\alpha^2-k^2)^{\frac{1}{2}}$, and written $p=(k^2+\mu^2)^{\frac{1}{2}}$, the root with $\operatorname{Im}p>0$ being implied. The first two terms in (A 8) are a minus and a plus function respectively. To split up the last two terms we use the functions $P_\pm(\alpha)$ defined in (A 4) giving, for example,

$$\frac{1}{(\alpha^2-k^2)^{\frac{1}{2}}}\frac{1}{\alpha-p} = \left(\frac{P_+(\alpha)-P_+(p)}{\alpha-p}\right)+\left(\frac{P_+(p)+P_-(\alpha)}{\alpha-p}\right), \tag{A 9}$$

in which the first term is a plus function, the second a minus function. Collecting up the terms, and using the identity $P_+(p)\equiv P_-(-p)$, we have

$$\frac{K_-'(\alpha)}{K_-(\alpha)} = \frac{1}{2(\alpha-p)}(1-\mu P_-(\alpha)-\mu P_+(p))+\frac{\mu}{2(\alpha+p)}(P_+(p)-P_-(\alpha)). \tag{A 10}$$

Now define a particular $K_{-}(\alpha)$ such that

$$\lim \alpha^{-\frac{1}{2}} K_{-}(\alpha) = 1,$$

as $\alpha \to \infty$ along the negative imaginary axis. Then integration and involution of (A 10) gives

$$K_{-}(\alpha) = (\alpha - p)^{\frac{1}{2}} \left(\frac{\alpha + p}{\alpha - p}\right)^{\frac{1}{2}\mu P_{+}(p)} \exp\left\{\mu \int_{\alpha}^{-i\infty} \frac{P_{-}(t)\, t\, dt}{t^2 - \mu^2 - k^2}\right\}. \tag{A 11}$$

Finally we require an estimate of $K_{-}(-k\cos\theta_0)$ for large values of μ/k. The asymptotic evaluation of (A 11) is complicated, though reasonably straightforward, and we merely quote the result that

$$K_{-}(-k\cos\theta_0) \sim \mu^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}. \tag{A 12}$$

We note also that (A 5) can be obtained directly from (A 11), but the iteration procedure is both simpler and capable of generalization to cases where the exact factorization is either impossible to obtain or hopelessly complicated.

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